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Abstract
We consider the six-vertex model on an $N \times N$ square lattice with the domain wall boundary conditions. Boundary one-point correlation functions of the model are expressed as determinants of $N \times N$ matrices, generalizing the known result for the partition function. In the free fermion case explicit answers are obtained. The introduced correlation functions are closely related to the problem of enumeration of alternating sign matrices and domino tilings.

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1. Introduction

The six-vertex model was studied for both periodic [1, 2] and fixed boundary conditions [3–6]. The particular example of fixed boundary conditions [7] of the model on an $N \times N$ square lattice is the so-called domain wall boundary conditions (DWBC) [8]. Under special restrictions on the vertex weights this model is related to the enumeration of alternating sign matrices [9, 10] and domino tilings of Aztec diamonds [9].

The model with DWBC originally appeared in the context of investigation of norms of the Bethe states in the framework of the quantum inverse scattering method (QISM) [11]. In the last decade the six-vertex model with DWBC has found interesting applications in different fields of physics and mathematics [12–16] due to the results of the papers [17, 18], where the determinant formula for the partition function has been obtained and proved. This determinant formula allowed several problems to be solved in combinatorics [10] which were standing for a long time [19].

A wide range of problems such as refined enumeration of alternating sign matrices (ASM) [20] and the arctic circle theorem [21, 22] can be solved only if the correlation functions of the model are known. In general, the calculation of the correlation functions is a more complicated problem than that of the partition function. Additional difficulties may arise due to the lack of translation invariance caused by the fixed boundary conditions.

In this paper we will consider two kinds of one-point boundary correlation functions of the six-vertex model with DWBC. The function of the first kind, $G_N^{(M)}$, is the local state probability on the boundary vertical edge and it may be called 'boundary spontaneous polarization'.
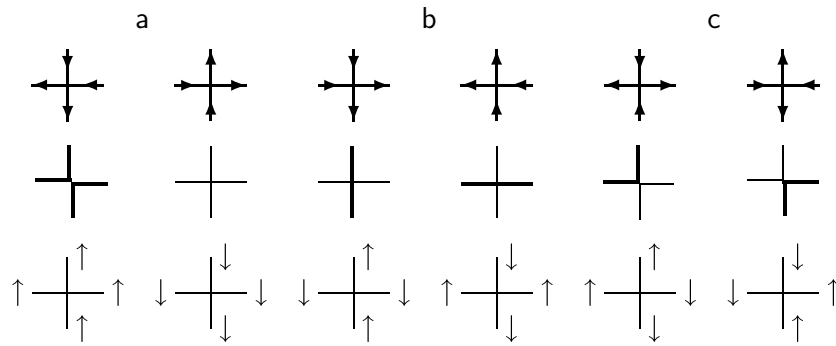
Figure 1. The six allowed types of vertices in terms of arrows (first row), lines (second row) and spins (third row).
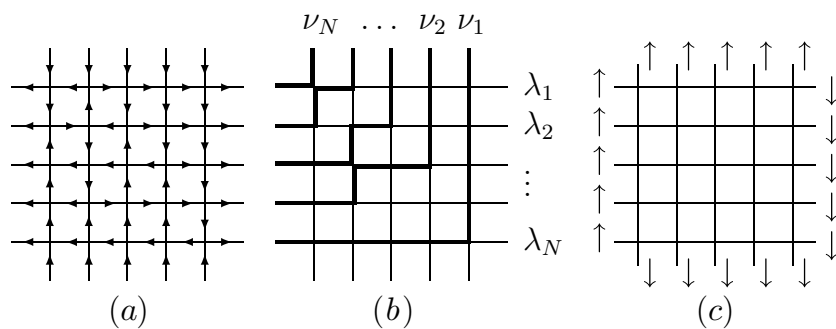

Figure 2. One of the possible configurations in the model with DWBC in terms of (*a*) arrows and (*b*) lines. (*c*) DWBC in terms of spins.

The function of the second kind, $H_N^{(M)}$, describes the probability of the vertex being in a specific state. For the model on an $N \times N$ square lattice we obtain representations for these correlation functions as determinants of $N \times N$ matrices. These correlation functions are the generalization of a boundary correlation function considered in [23].

There are three convenient ways for the description of the six-vertex model: (i) in terms of arrows pointing into and away from each vertex; (ii) in terms of lines flowing through the vertices; (iii) in terms of spins on the edges. The six types of vertices allowed in the model are plotted in figure 1. A statistical (vertex) weight corresponds to each type of vertex. We consider the six-vertex model with the vertex weights being invariant under the simultaneous reversal of all arrows. Hence, there are three different vertex weights, a, b and c (see figure 1).

The domain wall boundary conditions imply that all arrows on the top and bottom of the lattice are pointing inwards while all arrows on the left and right boundaries are pointing outwards (see figure 2(*a*)). It means that the solid lines flow from the top of the lattice to the left boundary (see figure 2(*b*)). In figures 2(*a*) and (*b*) one of the possible arrangements is presented. The domain wall boundary conditions in terms of spins are shown in figure 2(*c*).

In the inhomogeneous model the vertex weights a, b and c are site dependent. To introduce this dependence we will use two sets of the variables $\{\lambda_\alpha\}$ and $\{\nu_k\}$ that are in one-to-one correspondence with the set of lines. The rows will be enumerated by Greek indices $\alpha = 1, \ldots, N$ and the variable λ_α corresponds to the αth row; the columns will be

enumerated by Latin indices $k = 1, \ldots, N$ and the variable ν_k corresponds to the kth column. This correspondence is shown in figure 2(b). Each statistical weight associated with the vertex lying at the intersection of the αth row and kth column will depend on the pair of variables (λ_α, ν_k). The parametrization that allows one to apply QISM is

$$\mathsf{a}(\lambda_\alpha, \nu_k) = \sinh(\lambda_\alpha - \nu_k + \eta)$$
$$\mathsf{b}(\lambda_\alpha, \nu_k) = \sinh(\lambda_\alpha - \nu_k - \eta) \tag{1}$$
$$\mathsf{c}(\lambda_\alpha, \nu_k) = \sinh 2\eta.$$

In the homogeneous limit all $\lambda_\alpha \to \lambda$ and all $\nu_k \to \nu$. All positive values of the vertex weights, up to an overall scaling transformation, may be obtained by choosing both $\lambda - \nu$ and η either real or pure imaginary.

   Our calculations are based on the quantum inverse scattering method (QISM), reviewed briefly in section 2. In section 3 we derive the reduction formulae for the boundary correlation functions $G_N^{(M)}$ and $H_N^{(M)}$. The recursion relation for the partition function follows from these formulae as a particular case. The determinant representation for $G_N^{(M)}$ and $H_N^{(M)}$ is obtained from these reduction formulae in section 4. In the free fermion case the homogeneous limit for these boundary correlation functions is calculated explicitly in section 5, while the general case of the homogeneous limit is considered in section 6.

2. Formulation of the model within the QISM formalism

To apply the quantum inverse scattering method [11] we use the spin description of the model. With each vertical line (column) and horizontal line (row) one associates the space $\mathbb{C}^2$, with spin-up and spin-down states forming a natural basis in this space. The total space of the vertical lines is $\mathcal{V} = (\mathbb{C}^2)^{\otimes N}$ and the total space of the horizontal lines is $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$. With each vertex of the lattice one associates an operator acting in the full space $\mathcal{V} \otimes \mathcal{H}$. This operator is called the L-operator and it acts nontrivially only in a single horizontal space $\mathbb{C}^2$ and in a single vertical space $\mathbb{C}^2$, while in all other spaces it acts as the unity operator. To distinguish the spaces in which the L-operator acts nontrivially one can label it as $\mathsf{L}_{\alpha k}$ and associate it with the vertex being the intersection of the αth row and kth column. The matrix elements of the L-operator (which is a $2^{2N} \times 2^{2N}$ matrix) are either zeros or functions $\mathsf{a}(\lambda_\alpha, \nu_k)$, $\mathsf{b}(\lambda_\alpha, \nu_k)$, $\mathsf{c}(\lambda_\alpha, \nu_k)$, defined in (1). Hence, the L-operator $\mathsf{L}_{\alpha k}$ is a function of λ_α and ν_k, $\mathsf{L}_{\alpha k}(\lambda_\alpha, \nu_k)$. Since the L-operator acts nontrivially only in the direct product of a pair of two-dimensional spaces, all its elements may be written in the compact form as

$$\mathsf{L}_{\alpha k}(\lambda_\alpha, \nu_k) = \begin{pmatrix} \mathsf{a}(\lambda_\alpha, \nu_k) & 0 & 0 & 0 \\ 0 & \mathsf{b}(\lambda_\alpha, \nu_k) & \mathsf{c}(\lambda_\alpha, \nu_k) & 0 \\ 0 & \mathsf{c}(\lambda_\alpha, \nu_k) & \mathsf{b}(\lambda_\alpha, \nu_k) & 0 \\ 0 & 0 & 0 & \mathsf{a}(\lambda_\alpha, \nu_k) \end{pmatrix}_{[\alpha k]}. \tag{2}$$

This is the matrix with respect to the αth copy of $\mathbb{C}^2$ in $\mathcal{H}$ and the kth copy in $\mathcal{V}$, with the matrix elements being trivial matrices in the rest of the copies of $\mathbb{C}^2$ in $\mathcal{V}$ and in $\mathcal{H}$. One can write the L-operator in an alternative form with separated 'horizontal' and 'vertical' spaces, namely, as a matrix with respect to the αth copy of $\mathbb{C}^2$ in $\mathcal{H}$ with the operator matrix elements acting nontrivially only in the kth copy of $\mathbb{C}^2$ in $\mathcal{V}$

$$\mathsf{L}_{\alpha k}(\lambda_\alpha, \nu_k) = \begin{pmatrix} \sinh\left(\lambda_\alpha - \nu_k + \eta\sigma_k^z\right) & \sigma_k^- \sinh 2\eta \\ \sigma_k^+ \sinh 2\eta & \sinh\left(\lambda_\alpha - \nu_k - \eta\sigma_k^z\right) \end{pmatrix}_{[\alpha]} \tag{3}$$

where $\sigma_k^z, \sigma_k^\pm = \frac{1}{2}\left(\sigma_k^x \pm i\sigma_k^y\right)$ are Pauli matrices.

The main object of QISM is the 'vertical' monodromy matrix $\mathsf{T}_\alpha(\lambda_\alpha)$ which is defined as the ordered matrix product of the L-operators along the αth horizontal line:

$$\mathsf{T}_\alpha(\lambda_\alpha) = \mathsf{L}_{\alpha N}(\lambda_\alpha, \nu_N) \cdots \mathsf{L}_{\alpha 1}(\lambda_\alpha, \nu_1) = \begin{pmatrix} A(\lambda_\alpha) & B(\lambda_\alpha) \\ C(\lambda_\alpha) & D(\lambda_\alpha) \end{pmatrix}_{[\alpha]}. \tag{4}$$

All entries of the monodromy matrix $\mathsf{T}_\alpha(\lambda_\alpha)$ are operators acting in $\mathcal{V}$ and they depend on the variables $\nu_1, \ldots, \nu_N$, i.e. $A(\lambda) = A\big(\lambda; \{\nu_k\}_{k=1}^N\big)$, etc. Obviously, instead of the 'vertical' monodromy matrix one may use the 'horizontal' one, which is the ordered product of the L-operators along the vertical line.

The quantum inverse scattering method is based on the intertwining relation for the L-operators:

$$\mathsf{R}_{\alpha\beta}(\lambda_\alpha, \lambda_\beta)\mathsf{L}_{\alpha k}(\lambda_\alpha, \nu_k)\mathsf{L}_{\beta k}(\lambda_\beta, \nu_k) = \mathsf{L}_{\beta k}(\lambda_\beta, \nu_k)\mathsf{L}_{\alpha k}(\lambda_\alpha, \nu_k)\mathsf{R}_{\alpha\beta}(\lambda_\alpha, \lambda_\beta) \qquad \alpha \neq \beta. \tag{5}$$

The R-matrix $\mathsf{R}_{\alpha\beta}(\lambda, \lambda')$ acts nontrivially in the direct product of the αth and βth horizontal spaces and is given by

$$\mathsf{R}_{\alpha\beta}(\lambda, \lambda') = \begin{pmatrix} f(\lambda', \lambda) & 0 & 0 & 0 \\ 0 & 1 & g(\lambda', \lambda) & 0 \\ 0 & g(\lambda', \lambda) & 1 & 0 \\ 0 & 0 & 0 & f(\lambda', \lambda) \end{pmatrix}_{[\alpha\beta]} \tag{6}$$

where the functions $f(\lambda', \lambda)$ and $g(\lambda', \lambda)$ are

$$f(\lambda', \lambda) = \frac{\sinh(\lambda - \lambda' + 2\eta)}{\sinh(\lambda - \lambda')} \qquad g(\lambda', \lambda) = \frac{\sinh 2\eta}{\sinh(\lambda - \lambda')}. \tag{7}$$

This is the so-called trigonometric R-matrix [11], which satisfies the Yang–Baxter equation

$$\mathsf{R}_{\alpha\beta}(\lambda_\alpha, \lambda_\beta)\mathsf{R}_{\alpha\gamma}(\lambda_\alpha, \lambda_\gamma)\mathsf{R}_{\beta\gamma}(\lambda_\beta, \lambda_\gamma) = \mathsf{R}_{\beta\gamma}(\lambda_\beta, \lambda_\gamma)\mathsf{R}_{\alpha\gamma}(\lambda_\alpha, \lambda_\gamma)\mathsf{R}_{\alpha\beta}(\lambda_\alpha, \lambda_\beta)$$
$$\alpha \neq \beta \neq \gamma. \tag{8}$$

Due to relation (5) and commutativity of the matrix elements of the L-operator (3) at different lattice sites one has the intertwining relation for the monodromy matrix

$$\mathsf{R}_{\alpha\beta}(\lambda_\alpha, \lambda_\beta)\mathsf{T}_\alpha(\lambda_\alpha)\mathsf{T}_\beta(\lambda_\beta) = \mathsf{T}_\beta(\lambda_\beta)\mathsf{T}_\alpha(\lambda_\alpha)\mathsf{R}_{\alpha\beta}(\lambda_\alpha, \lambda_\beta) \qquad \alpha \neq \beta. \tag{9}$$

Equation (9) defines the commutation relations for the operators entering the monodromy matrix. The complete list of these relations can be found, e.g., in [11]. For our purposes we need only two of them:

$$A(\lambda)B(\lambda') = f(\lambda, \lambda')B(\lambda')A(\lambda) + g(\lambda', \lambda)B(\lambda)A(\lambda')$$
$$B(\lambda)B(\lambda') = B(\lambda')B(\lambda). \tag{10}$$

As the generating vector in the space $\mathcal{V}$, it is convenient to use the state either with all spins up or with all spins down:

$$|\Uparrow\rangle = \otimes_{k=1}^N |\uparrow\rangle_k = \otimes_{k=1}^N \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{[k]} \qquad |\Downarrow\rangle = \otimes_{k=1}^N |\downarrow\rangle_k = \otimes_{k=1}^N \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{[k]}. \tag{11}$$

These vectors are annihilated by the operators $C(\lambda)$ and $B(\lambda)$, respectively,

$$C(\lambda)|\Uparrow\rangle = 0 \qquad B(\lambda)|\Downarrow\rangle = 0 \tag{12}$$

and they are eigenvectors of the operators $A(\lambda)$ and $D(\lambda)$,

$$A(\lambda)|\Uparrow\rangle = a(\lambda)|\Uparrow\rangle \qquad D(\lambda)|\Uparrow\rangle = d(\lambda)|\Uparrow\rangle$$
$$A(\lambda)|\Downarrow\rangle = d(\lambda)|\Downarrow\rangle \qquad D(\lambda)|\Downarrow\rangle = a(\lambda)|\Downarrow\rangle \tag{13}$$

where the functions $a(\lambda)$ and $d(\lambda)$ are equal to

$$a(\lambda) = \prod_{k=1}^{N} \sinh(\lambda - \nu_k + \eta) \qquad \text{and} \qquad d(\lambda) = \prod_{k=1}^{N} \sinh(\lambda - \nu_k - \eta) \qquad (14)$$

respectively. The vectors $\langle\Uparrow|$ and $\langle\Downarrow|$, dual to (11), are eigenvectors of the operators $A(\lambda)$ and $D(\lambda)$ with the same eigenvalues as in equations (13) while instead of equations (12) one has

$$\langle\Uparrow|B(\lambda) = 0 \qquad \langle\Downarrow|C(\lambda) = 0. \qquad (15)$$

Consider vectors generated by the multiple action of operators $B(\lambda_\alpha)$ on the state $|\Uparrow\rangle$:

$$B(\lambda_M) \cdots B(\lambda_1)|\Uparrow\rangle \qquad M \leqslant N. \qquad (16)$$

The result of the action of the operator $A(\lambda)$ on vector (16) follows from commutation relations (10):

$$A(\lambda) \prod_{\alpha=1}^{M} B(\lambda_\alpha)|\Uparrow\rangle = \Lambda \prod_{\alpha=1}^{M} B(\lambda_\alpha)|\Uparrow\rangle + \sum_{\beta=1}^{M} \Lambda_\beta B(\lambda) \prod_{\substack{\alpha=1 \\ \alpha\neq\beta}}^{M} B(\lambda_\alpha)|\Uparrow\rangle \qquad (17)$$

where the coefficients Λ and Λ_β are

$$\Lambda = a(\lambda) \prod_{\gamma=1}^{M} f(\lambda, \lambda_\gamma) \qquad \Lambda_\beta = a(\lambda_\beta) g(\lambda_\beta, \lambda) \prod_{\substack{\gamma=1 \\ \gamma\neq\beta}}^{M} f(\lambda_\beta, \lambda_\gamma). \qquad (18)$$

The partition function $Z_N = Z_N(\lambda_1, \ldots, \lambda_N; \nu_1, \ldots, \nu_N)$ of the model on an $N \times N$ square lattice with DWBC is obtained by summation over the contributions of all possible spin configurations. The contribution of each configuration is equal to the product of all vertex weights of this configuration. In terms of QISM the partition function may be represented as

$$Z_N = \left(\overset{N}{\underset{\alpha=1}{\otimes}}_{\alpha}\langle\uparrow|\right) \otimes \left(\overset{N}{\underset{k=1}{\otimes}}_{k}\langle\downarrow|\right) T_N(\lambda_N) \cdots T_1(\lambda_1) \left(\overset{N}{\underset{k=1}{\otimes}}|\uparrow\rangle_k\right) \otimes \left(\overset{N}{\underset{\alpha=1}{\otimes}}|\downarrow\rangle_\alpha\right). \qquad (19)$$

The boundary conditions on the left and the right boundaries extract the operator $B(\lambda_\alpha)$ from each matrix $T_\alpha(\lambda_\alpha)$. The boundary conditions on the top (bottom) of the lattice correspond to the vector $|\Uparrow\rangle$ ($\langle\Downarrow|$), respectively. Hence, the partition function can be written in the form

$$Z_N = \langle\Downarrow|B(\lambda_N) \cdots B(\lambda_1)|\Uparrow\rangle. \qquad (20)$$

Due to relation (10) the order of operators $B(\lambda_\alpha)$ in the product is not essential.

The determinant representation for the partition function Z_N was obtained in the papers [17, 18] and has the form

$$Z_N = \frac{\prod_{\alpha=1}^{N} \prod_{k=1}^{N} \sinh(\lambda_\alpha - \nu_k + \eta) \sinh(\lambda_\alpha - \nu_k - \eta)}{\prod_{1\leqslant\alpha<\beta\leqslant N} \sinh(\lambda_\beta - \lambda_\alpha) \prod_{1\leqslant k<j\leqslant N} \sinh(\nu_k - \nu_j)} \det_N \mathcal{Z}. \qquad (21)$$

The entries of the matrix $\mathcal{Z}$ are given by

$$\mathcal{Z}_{\alpha k} = \phi(\lambda_\alpha, \nu_k) \qquad \alpha, k = 1, \ldots, N \qquad (22)$$

where the function $\phi(\lambda, \nu)$ is

$$\phi(\lambda, \nu) = \frac{\sinh 2\eta}{\sinh(\lambda - \nu + \eta) \sinh(\lambda - \nu - \eta)}. \qquad (23)$$

The proof of determinant representation (21) based exclusively on commutation relations (10) is given in section 4.

3. Boundary correlation functions

In the present paper we consider two kinds of correlation functions describing the local state probabilities at the boundary. The first correlation function describes the probability of the absence of a vertical solid line between the $(M + 1)$th and Mth rows of the first column and is known as 'boundary spontaneous polarization'. In terms of QISM it is the one-point correlation function of the local spin projector $q_1 = \frac{1}{2}\left(1 - \sigma_1^z\right)$ on the spin-down state, and it can be written as

$$G_N^{(M)} = Z_N^{-1}\langle\Downarrow|B(\lambda_N)\cdots B(\lambda_{M+1})q_1 B(\lambda_M)\cdots B(\lambda_1)|\Uparrow\rangle. \tag{24}$$

The second correlation function describes the probability that the solid line on the first column turns to the left just on the Mth row,

$$H_N^{(M)} = Z_N^{-1}\langle\Downarrow|B(\lambda_N)\cdots B(\lambda_{M+1})q_1 B(\lambda_M)p_1 B(\lambda_{M-1})\cdots B(\lambda_1)|\Uparrow\rangle \tag{25}$$

where p_1 is the projector on the spin-up state, $p_1 = \frac{1}{2}\left(1 + \sigma_1^z\right)$. Since $p_1 + q_1 = I$, these correlation functions are related to each other as follows:

$$G_N^{(M)} = H_N^{(M)} + H_N^{(M-1)} + \cdots + H_N^{(1)} \tag{26}$$

$$H_N^{(M)} = G_N^{(M)} - G_N^{(M-1)}. \tag{27}$$

However, it is easier to calculate them from definitions (24) and (25) independently.

   In this section we express the correlation functions of the model on an $N \times N$ square lattice through the sum over partition functions of the models on $(N - 1) \times (N - 1)$ square sublattices. We will call the corresponding formulae the 'reduction formulae'. The derivation of these formulae is based exclusively on commutation relations (10). Since $G_N^{(N)} = 1$, in the particular case $M = N$ the reduction formula for $G_N^{(M)}$ turns into the recursion relation for the partition function. In the next section we will prove that determinant representation (21) is the solution of this recursion relation. The determinant representation for the correlation functions can then be obtained by substituting expression (21) in the reduction formulae, which makes our algebraic approach self-contained.

   To derive the reduction formulae for the correlation functions we rewrite them in a form suitable for applying commutation relations (10). Let us decompose the monodromy matrix $\mathsf{T}_\alpha(\lambda_\alpha)$ into the matrix product of two monodromy matrices in the αth space:

$$\mathsf{T}_\alpha(\lambda_\alpha) = \mathsf{T}_{\alpha 2}(\lambda_\alpha)\mathsf{T}_{\alpha 1}(\lambda_\alpha). \tag{28}$$

This decomposition of the monodromy matrix is known as the 'two-site model' [11]. In our case we choose these matrices defined as follows:

$$\mathsf{T}_{\alpha 2}(\lambda_\alpha) = \mathsf{L}_{\alpha N}(\lambda_\alpha, \nu_N)\cdots\mathsf{L}_{\alpha 2}(\lambda_\alpha, \nu_2) = \begin{pmatrix} A_2(\lambda_\alpha) & B_2(\lambda_\alpha) \\ C_2(\lambda_\alpha) & D_2(\lambda_\alpha) \end{pmatrix}_{[\alpha]}$$
$$\mathsf{T}_{\alpha 1}(\lambda_\alpha) = \mathsf{L}_{\alpha 1}(\lambda_\alpha, \nu_1) = \begin{pmatrix} A_1(\lambda_\alpha) & B_1(\lambda_\alpha) \\ C_1(\lambda_\alpha) & D_1(\lambda_\alpha) \end{pmatrix}_{[\alpha]}. \tag{29}$$

The matrix elements of $\mathsf{T}_{\alpha 2}(\lambda)$ commute with the matrix elements of $\mathsf{T}_{\alpha 1}(\lambda)$ since they are operators that act nontrivially in different spaces. The entries of $\mathsf{T}_{\alpha 1}(\lambda)$ act nontrivially in the first 'vertical' space and depend on the 'vertical' variable ν_1, while the entries of $\mathsf{T}_{\alpha 2}(\lambda)$ act nontrivially in the rest of the $N - 1$ 'vertical' spaces and depend on the 'vertical' variables $\nu_2, \ldots, \nu_N$. Each set of operators entering the monodromy matrices satisfies the commutation relations given by (9) and, in particular, (10). It should be noted that the generating vectors $|\Uparrow\rangle(|\Downarrow\rangle)$ can be represented as the direct product of two generating vectors,

e.g., $|\Uparrow\rangle = |\Uparrow_2\rangle \otimes |\Uparrow_1\rangle$), where $|\Uparrow_2\rangle = \otimes_{k=2}^{N}|\uparrow\rangle_k$ and $|\Uparrow_1\rangle \equiv |\uparrow\rangle_1$. The properties of both the states $|\Uparrow_2\rangle$ $(|\Downarrow_2\rangle)$ and $|\Uparrow_1\rangle$ $(|\Downarrow_1\rangle)$ are the same as those of $|\Uparrow\rangle$ $(|\Downarrow\rangle)$; see the previous section. The eigenvalue of, e.g., operator $A_2(\lambda)$ on the state $|\Uparrow_2\rangle$ is

$$A_2(\lambda)|\Uparrow_2\rangle = a_2(\lambda)|\Uparrow_2\rangle \qquad a_2(\lambda) = \prod_{k=2}^{N} \sinh(\lambda - v_k + \eta). \tag{30}$$

The decomposition of the operators $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ follows from definitions (28), (29) and (4). In particular, one has

$$B(\lambda) = A_2(\lambda)B_1(\lambda) + B_2(\lambda)D_1(\lambda). \tag{31}$$

The operators $B_1(\lambda)$ and $D_1(\lambda)$ are the corresponding entries of the L-operator (3) and they are given by

$$B_1(\lambda) = \sigma_1^{-} \sinh 2\eta \qquad D_1(\lambda) = \sinh\left(\lambda - v_1 - \eta\sigma_1^{z}\right). \tag{32}$$

Using formulae (31) and (32) one can reduce the problem of calculation of the scalar products on the right-hand sides of expressions (24) and (25) to the problem of calculation of the scalar products involving the operators $A_2(\lambda)$ and $B_2(\lambda)$ only. Now we are ready to derive the reduction formulae for the correlation functions.

We start with the derivation of the reduction formula for the function $H_N^{(M)}$ since this derivation is straightforward. Let us substitute expression (31) in (25) and calculate the scalar product with respect to $\langle\Downarrow_1|$ and $|\Uparrow_1\rangle$ using formula (32). We are left with the expression

$$H_N^{(M)} = Z_N^{-1} \sinh 2\eta \prod_{\alpha=1}^{M-1} \sinh(\lambda_\alpha - v_1 - \eta) \prod_{\alpha=M+1}^{N} \sinh(\lambda_\alpha - v_1 + \eta)$$
$$\times \langle\Downarrow_2|B_2(\lambda_N)\cdots B_2(\lambda_{M+1})A_2(\lambda_M)B_2(\lambda_{M-1})\cdots B_2(\lambda_1)|\Uparrow_2\rangle. \tag{33}$$

Applying (17) and taking into account (30), we reduce the scalar product in (33) to the sum over scalar products that involve only the operators B_2. Comparing these scalar products with expression (20), one immediately gets the following formula for the correlation function $H_N^{(M)}$:

$$H_N^{(M)} = Z_N^{-1} \sinh 2\eta \prod_{\alpha=1}^{M-1} \sinh(\lambda_\alpha - v_1 - \eta) \prod_{\alpha=M+1}^{N} \sinh(\lambda_\alpha - v_1 + \eta)$$
$$\times \sum_{\beta=1}^{M} a_2(\lambda_\beta)\frac{g(\lambda_\beta, \lambda_M)}{f(\lambda_\beta, \lambda_M)} \prod_{\substack{\gamma=1 \\ \gamma\neq\beta}}^{M} f(\lambda_\beta, \lambda_\gamma) Z_{N-1}\big(\{\lambda_\alpha\}_{\alpha=1,\alpha\neq\beta}^{N}; \{v_k\}_{k=2}^{N}\big). \tag{34}$$

In the derivation of this formula we have used the fact that the functions $f(\lambda', \lambda)$ and $g(\lambda', \lambda)$ defined in (7) satisfy the condition $g(\lambda, \lambda)/f(\lambda, \lambda) = 1$. This allows us to include the first term arising in (17) into the sum over β.

Equation (34) expresses the particular boundary correlation function of the model on an $N \times N$ square lattice as the sum over partition functions of the models on $(N-1) \times (N-1)$ square sublattices. It should be stressed that in the derivation of the formula we have used only the algebra of operators $A_2(\lambda)$ and $B_2(\lambda)$ given by commutation relations (10).

Let us turn now to the correlation function $G_N^{(M)}$ defined by expression (24). First, we substitute formulae (31) and (32) in (24). This gives

$$G_N^{(M)} = Z_N^{-1} \sinh 2\eta \sum_{\beta=1}^{M} \prod_{\alpha=1}^{\beta-1} \sinh(\lambda_\alpha - v_1 - \eta) \prod_{\alpha=\beta+1}^{N} \sinh(\lambda_\alpha - v_1 + \eta)$$
$$\times \langle\Downarrow_2|B_2(\lambda_N)\cdots B_2(\lambda_{\beta+1})A_2(\lambda_\beta)B_2(\lambda_{\beta-1})\cdots B_2(\lambda_1)|\Uparrow_2\rangle. \tag{35}$$

To obtain the representation of the correlation function $G_N^{(M)}$ as the sum over the partition functions of the models on $(N-1) \times (N-1)$ square sublattices one can substitute formula (17) in expression (35), but then the double sum will appear in the resulting expression. To represent the right-hand side of expression (35) as a single sum, as in formula (34), we choose another way. In expression (35) we pick up the term which contains the operator A_2 standing furthest to the left among the operators B_2, i.e. the term corresponding to $\beta = M$. Taking into account formula (17) it is easy to see that this term contains the following contribution, which corresponds to the first term on the right-hand side of (17):

$$
\langle \Downarrow_2 | B_2(\lambda_N) \cdots B_2(\lambda_{M+1}) A_2(\lambda_M) B_2(\lambda_{M-1}) \cdots B_2(\lambda_1) | \Uparrow_2 \rangle
$$
$$
= a_2(\lambda_M) \prod_{\gamma=1}^{M-1} f(\lambda_M, \lambda_\gamma) \langle \Downarrow_2 | B_2(\lambda_N) \cdots B_2(\lambda_{M+1}) B_2(\lambda_{M-1}) \cdots B_2(\lambda_1) | \Uparrow_2 \rangle
$$
$$
+ \text{other terms} \tag{36}
$$

where the 'other terms' are the terms with scalar products involving the operator $B_2(\lambda_M)$. Substituting equation (36) in (35) and applying formula (17) to the rest of the terms in (35), we obtain that

$$
G_N^{(M)} = Z_N^{-1} \prod_{\alpha=1}^{M} \sinh(\lambda_\alpha - \nu_1 - \eta) \prod_{\alpha=M+1}^{N} \sinh(\lambda_\alpha - \nu_1 + \eta) \left(\frac{a_2(\lambda_M) \sinh 2\eta}{\sinh(\lambda_M - \nu_1 - \eta)} \right.
$$
$$
\times \prod_{\gamma=1}^{M-1} f(\lambda_M, \lambda_\gamma) \langle \Downarrow_2 | B_2(\lambda_N) \cdots B_2(\lambda_{M+1}) B_2(\lambda_{M-1}) \cdots B_2(\lambda_1) | \Uparrow_2 \rangle
$$
$$
\left. + \text{other terms} \right) \tag{37}
$$

where the 'other terms' are again the terms, different to those in (36), with scalar products involving the operator $B_2(\lambda_M)$. Thus, the contribution written explicitly in (37) is the only possible one which does not contain the operator $B_2(\lambda_M)$ in the scalar product. Obviously, this contribution is symmetric under the permutations of the elements in the set $\lambda_1, \ldots, \lambda_{M-1}$. At the same time the correlation function $G_N^{(M)}$ is symmetric under the permutations of the elements in the set $\lambda_1, \ldots, \lambda_M$, due to the commutativity of the operators $B_2(\lambda_\alpha)$. It follows immediately from these symmetry considerations that the whole expression given in the parentheses in (37) is just the sum over the cyclic permutations of the elements in the set $\lambda_1, \ldots, \lambda_M$ of the term that is written explicitly on the right-hand side of equation (37). Therefore, the following representation for the correlation function $G_N^{(M)}$ is valid:

$$
G_N^{(M)} = Z_N^{-1} \prod_{\alpha=1}^{M} \sinh(\lambda_\alpha - \nu_1 - \eta) \prod_{\alpha=M+1}^{N} \sinh(\lambda_\alpha - \nu_1 + \eta)
$$
$$
\times \sum_{\beta=1}^{M} \frac{a_2(\lambda_\beta) \sinh 2\eta}{\sinh(\lambda_\beta - \nu_1 - \eta)} \prod_{\substack{\gamma=1 \\ \gamma \neq \beta}}^{M} f(\lambda_\beta, \lambda_\gamma) Z_{N-1}\left(\{\lambda_\alpha\}_{\alpha=1, \alpha \neq \beta}^{N}; \{\nu_k\}_{k=2}^{N} \right). \tag{38}
$$

The procedure described above is similar to the derivation of equation (17) from commutation relations (10); see, e.g., [11].

The important point to be discussed now is the meaning of representation (38) at $M = N$. Since the left-hand side of equation (38) is equal to one at $M = N$, it turns into the recursion relation on the partition function. In this relation Z_N is expressed through the sum over the

partition functions Z_{N-1}:

$$Z_N\big(\{\lambda_\alpha\}_{\alpha=1}^N; \{\nu_k\}_{k=1}^N\big) = \sinh 2\eta \sum_{\beta=1}^N \prod_{\substack{\alpha=1 \\ \alpha \neq \beta}}^N \sinh(\lambda_\alpha - \nu_1 - \eta) \prod_{k=2}^N \sinh(\lambda_\beta - \nu_k + \eta)$$

$$\times \prod_{\substack{\gamma=1 \\ \gamma \neq \beta}}^N f(\lambda_\beta, \lambda_\gamma) Z_{N-1}\big(\{\lambda_\alpha\}_{\alpha=1,\alpha\neq\beta}^N; \{\nu_k\}_{k=2}^N\big). \tag{39}$$

This formula was obtained by slightly different technique in [24]. Due to the symmetry of Z_N with respect to the permutations of the variables $\{\nu_k\}_{k=1}^N$, one can rewrite this recursion relation in a general form:

$$Z_N\big(\{\lambda_\alpha\}_{\alpha=1}^N; \{\nu_k\}_{k=1}^N\big) = \sinh 2\eta \sum_{\beta=1}^N \prod_{\substack{\alpha=1 \\ \alpha \neq \beta}}^N \sinh(\lambda_\alpha - \nu_j - \eta) \prod_{\substack{k=1 \\ k\neq j}}^N \sinh(\lambda_\beta - \nu_k + \eta)$$

$$\times \prod_{\substack{\gamma=1 \\ \gamma \neq \beta}}^N f(\lambda_\beta, \lambda_\gamma) Z_{N-1}\big(\{\lambda_\alpha\}_{\alpha=1,\alpha\neq\beta}^N; \{\nu_k\}_{k=1,k\neq j}^N\big) \qquad j = 1, \dots, N.$$

$$\tag{40}$$

Relations (39) and (40) are valid for arbitrary values of the variables $\{\lambda_\alpha\}_{\alpha=1}^N$ and $\{\nu_k\}_{k=1}^N$. Iterating relation (40) $N-1$ times, with the initial condition $Z_1 = \sinh 2\eta$, one gets the answer as the sum over all permutations P of $\{\lambda_\alpha\}_{\alpha=1}^N$ (cf (3.17) of [7])

$$Z_N = (\sinh 2\eta)^N \sum_P \prod_{1 \leqslant \alpha < \beta \leqslant N} f\big(\lambda_{P_\alpha}, \lambda_{P_\beta}\big) \sinh\big(\lambda_{P_\beta} - \nu_\alpha - \eta\big) \sinh\big(\lambda_{P_\alpha} - \nu_\beta + \eta\big). \tag{41}$$

On the other hand, one can show that determinant representation (21) for the partition function is the solution of recursion relation (40). The determinant solution of equation (40) leads to the determinant representations for the boundary correlation functions.

4. Determinant representations for boundary correlation functions

We begin this section with the proof that determinant formula (21) for the partition function solves recursion relation (40). The determinant representation for the boundary correlation functions will follow then from reduction formulae (34) and (38), which can be viewed as generalizations of the recursion relation for the partition function.

Let us give the proof for $j = 1$ in (40). This case is given by relation (39). For the other values of j the proof is essentially the same. Consider the left-hand side of relation (39) which is given by formula (21). The aim is to represent this expression in the form given by the right-hand side of relation (39). Since Z_N and Z_{N-1} are expressed through determinants of $N \times N$ and $(N-1) \times (N-1)$ matrices respectively, it is clear that relation (39) can be proved developing a determinant of some $N \times N$ matrix by the first column. To apply this approach one has to transform the determinant of the matrix $\mathcal{Z}$ in (21) first. Consider the function

$$g_N(\lambda) = \frac{\prod_{\gamma=1}^N \sinh(\lambda_\gamma - \lambda + 2\eta)}{\prod_{k=1}^N \sinh(\lambda - \nu_k - \eta)}. \tag{42}$$

For this function one has the identity

$$g_N(\lambda_\alpha) = \sum_{k=1}^N \Phi_k \phi(\lambda_\alpha, \nu_k) \qquad \alpha = 1, \dots, N \tag{43}$$

where the function $\phi(\lambda, \nu)$ is given by formula (23). Here the coefficients Φ_k are independent of α and are given as

$$\Phi_k = \frac{\prod_{\gamma=1}^{N} \sinh(\lambda_\gamma - \nu_k + \eta)}{\prod_{\substack{j=1 \\ j \neq k}}^{N} \sinh(\nu_k - \nu_j)} \qquad k = 1, \ldots, N. \tag{44}$$

Relation (43) is a short form of the identity

$$\frac{\prod_{\substack{\gamma=1 \\ \gamma \neq \alpha}}^{N} \sinh(\lambda_\gamma - \lambda_\alpha + 2\eta)}{\prod_{j=1}^{N} \sinh(\lambda_\alpha - \nu_j - \eta)} = \sum_{k=1}^{N} \frac{\prod_{\substack{\gamma=1 \\ \gamma \neq \alpha}}^{N} \sinh(\lambda_\gamma - \nu_k + \eta)}{\prod_{\substack{j=1 \\ j \neq k}}^{N} \sinh(\nu_k - \nu_j)} \frac{1}{\sinh(\lambda_\alpha - \nu_k - \eta)} \tag{45}$$

which may be proved, e.g., by induction, see also [25]. Relation (43) may be considered as the system of linear equations on the coefficients Φ_k with the right-hand side formed by the vector $g_N(\lambda_\alpha), \alpha = 1, \ldots, N$. Therefore, by Cramer's rule we obtain the connection formula for the determinants

$$\det_N \mathcal{Z} = \frac{1}{\Phi_1} \begin{vmatrix} g_N(\lambda_1) & \phi(\lambda_1, \nu_2) & \ldots & \phi(\lambda_1, \nu_N) \\ g_N(\lambda_2) & \phi(\lambda_2, \nu_2) & \ldots & \phi(\lambda_2, \nu_N) \\ \vdots & \vdots & \ddots & \vdots \\ g_N(\lambda_N) & \phi(\lambda_N, \nu_2) & \ldots & \phi(\lambda_N, \nu_N) \end{vmatrix}_N \tag{46}$$

where $\det_N \mathcal{Z}$ is given by formula (21). It is the transformed form of the determinant of the matrix $\mathcal{Z}$ which can be developed by the first column. Hence, one gets

$$\det_N \mathcal{Z} = \frac{1}{\Phi_1} \sum_{\beta=1}^{N} (-1)^{\beta-1} g_N(\lambda_\beta) \Delta_{N-1}^{(\beta)} \tag{47}$$

where $\Delta_{N-1}^{(\beta)}$ denotes the determinant of an $(N-1) \times (N-1)$ matrix

$$\Delta_{N-1}^{(\beta)} = \begin{vmatrix} \phi(\lambda_1, \nu_2) & \ldots & \phi(\lambda_1, \nu_N) \\ \vdots & \ddots & \vdots \\ \phi(\lambda_{\beta-1}, \nu_2) & \ldots & \phi(\lambda_{\beta-1}, \nu_N) \\ \phi(\lambda_{\beta+1}, \nu_2) & \ldots & \phi(\lambda_{\beta+1}, \nu_N) \\ \vdots & \ddots & \vdots \\ \phi(\lambda_N, \nu_2) & \ldots & \phi(\lambda_N, \nu_N) \end{vmatrix}_{N-1}. \tag{48}$$

Substituting determinant representation (21) for the partition functions Z_N and Z_{N-1} on the left- and right-hand sides of relation (39) respectively, and cancelling the resulting common factors, one gets exactly equation (47). This proves that the determinant representation (21) is the solution of recursion relation (39) for the partition function.

Let us turn now to the boundary correlation functions. Reduction formulae (34) and (38), obtained in the previous section, express them through the sum over partition functions of the models on $(N-1) \times (N-1)$ square sublattices. Using determinant representation (21) for the partition function one can then obtain the determinant representations for the boundary correlation functions.

Consider the correlation function $H_N^{(M)}$. Substituting expression (21) for Z_{N-1} in reduction formula (34) and extracting a general multiplier out of the sum over β, one gets the following expression:

$$H_N^{(M)} = Z_N^{-1} \sinh 2\eta \prod_{\alpha=1}^{M-1} \sinh(\lambda_\alpha - \nu_1 - \eta) \prod_{\alpha=M+1}^{N} \sinh(\lambda_\alpha - \nu_1 + \eta)$$

$$\times \frac{\prod_{\alpha=1}^{N} \prod_{k=2}^{N} \sinh(\lambda_\alpha - \nu_k + \eta) \sinh(\lambda_\alpha - \nu_k - \eta)}{\prod_{1 \leqslant \alpha < \beta \leqslant N} \sinh(\lambda_\beta - \lambda_\alpha) \prod_{2 \leqslant k < j \leqslant N} \sinh(\nu_k - \nu_j)}$$

$$\times \sum_{\beta=1}^{M} \frac{\prod_{\gamma=1}^{M-1} \sinh(\lambda_\gamma - \lambda_\beta + 2\eta) \prod_{\alpha=M+1}^{N} \sinh(\lambda_\gamma - \lambda_\beta)}{\prod_{k=2}^{N} \sinh(\lambda_\beta - \nu_k - \eta)} (-1)^{\beta-1} \Delta_{N-1}^{(\beta)} \quad (49)$$

where the quantity $\Delta_{N-1}^{(\beta)}$ is defined by expression (48). Clearly, the sum over β in formula (49) is nothing but the determinant of some $N \times N$ matrix developed by the first column. Only the M first entries in this column are not equal to zero, so there are M terms in the sum. Taking this into account and using expression (21) for Z_N, we finally obtain the following determinant representation for the correlation function $H_N^{(M)}$:

$$H_N^{(M)} = \frac{\sinh 2\eta \prod_{k=2}^{N} \sinh(\nu_1 - \nu_k)}{\prod_{\alpha=1}^{M} \sinh(\lambda_\alpha - \nu_1 + \eta) \prod_{\alpha=M}^{N} \sinh(\lambda_\alpha - \nu_1 - \eta)} \frac{\det_N \mathcal{H}}{\det_N \mathcal{Z}} \quad (50)$$

where the entries of the $N \times N$ matrix $\mathcal{H}$ are given by

$$\mathcal{H}_{\alpha 1} = h_M(\lambda_\alpha) \qquad \mathcal{H}_{\alpha k} = \phi(\lambda_\alpha, \nu_k) \qquad k = 2, \dots, N. \quad (51)$$

The function $h_M(\lambda)$ is equal to

$$h_M(\lambda) = \frac{\prod_{\gamma=1}^{M-1} \sinh(\lambda_\gamma - \lambda + 2\eta) \prod_{\gamma=M+1}^{N} \sinh(\lambda_\gamma - \lambda)}{\prod_{k=2}^{N} \sinh(\lambda - \nu_k - \eta)}. \quad (52)$$

The points $\lambda_{M+1}, \dots, \lambda_N$ are zeros of the function $h_M(\lambda)$; hence, the last $N - M$ entries in the first column of the matrix $\mathcal{H}$ are equal to zero. Note that the prefactor in (50) is, in fact, equal to $\phi(\lambda_M, \nu_1)/h_M(\nu_1 + \eta)$.

Consider now the correlation function $G_N^{(M)}$. Substituting expression (21) for Z_{N-1} in reduction formula (38) we arrive at the expression for the correlation function $G_N^{(M)}$ as the sum of $\Delta_{N-1}^{(\beta)}$. This expression is similar to (49) for the function $H_N^{(M)}$. Finally, we obtain the following determinant representation for the correlation function $G_N^{(M)}$:

$$G_N^{(M)} = \frac{\prod_{k=2}^{N} \sinh(\nu_1 - \nu_k)}{\prod_{\alpha=1}^{M} \sinh(\lambda_\alpha - \nu_1 + \eta) \prod_{\alpha=M+1}^{N} \sinh(\lambda_\alpha - \nu_1 - \eta)} \frac{\det_N \mathcal{G}}{\det_N \mathcal{Z}} \quad (53)$$

where the entries of the $N \times N$ matrix $\mathcal{G}$ are given by

$$\mathcal{G}_{\alpha 1} = g_M(\lambda_\alpha) \qquad \mathcal{G}_{\alpha k} = \phi(\lambda_\alpha, \nu_k) \qquad k = 2, \dots, N. \quad (54)$$

The function $g_M(\lambda)$ is equal to

$$g_M(\lambda) = \frac{\prod_{\gamma=1}^{M} \sinh(\lambda_\gamma - \lambda + 2\eta) \prod_{\gamma=M+1}^{N} \sinh(\lambda_\gamma - \lambda)}{\prod_{k=1}^{N} \sinh(\lambda - \nu_k - \eta)}. \quad (55)$$

At $M = N$ this function is exactly the function $g_N(\lambda)$ defined by formula (42). The points $\lambda_{M+1}, \dots, \lambda_N$ are zeros of the function $g_M(\lambda)$; hence, the last $N - M$ entries in the first column of the matrix $\mathcal{G}$ are equal to zero. By a direct check it is easy to be convinced that representations (50) and (53) satisfy relations (26) and (27).

Determinant representations (50) and (53) for the boundary correlation functions $H_N^{(M)}$ and $G_N^{(M)}$ are the main results of the present paper.

5. The free fermion case

The 'free fermion' condition implies the following restriction on the vertex weights:

$$\mathsf{a}^2(\lambda_\alpha, \nu_k) + \mathsf{b}^2(\lambda_\alpha, \nu_k) = \mathsf{c}^2(\lambda_\alpha, \nu_k) \qquad \alpha, k = 1, \ldots, N. \tag{56}$$

This equality is satisfied if we put $\eta = \mathrm{i}\frac{\pi}{4}$ in (1). It is convenient to change the variables $\lambda_\alpha \to \mathrm{i}\lambda_\alpha$, $\nu_k \to \mathrm{i}\nu_k$, and after the rescaling $\mathsf{a} \to -\mathrm{i}\mathsf{a}$, $\mathsf{b} \to -\mathrm{i}\mathsf{b}$, $\mathsf{c} \to -\mathrm{i}\mathsf{c}$ one gets the following parametrization of the vertex weights:

$$\begin{aligned}
\mathsf{a}(\lambda_\alpha, \nu_k) &= \sin\left(\lambda_\alpha - \nu_k + \tfrac{\pi}{4}\right) \\
\mathsf{b}(\lambda_\alpha, \nu_k) &= \sin\left(\lambda_\alpha - \nu_k - \tfrac{\pi}{4}\right) \\
\mathsf{c}(\lambda_\alpha, \nu_k) &= 1.
\end{aligned} \tag{57}$$

Since under the condition $\eta = \mathrm{i}\frac{\pi}{4}$ the determinant in equation (21) becomes the Cauchy determinant, the partition function for the free fermion case can be evaluated explicitly:

$$Z_N\left(\{\lambda_\alpha\}_{\alpha=1}^N; \{\nu_k\}_{k=1}^N\right) = \prod_{1 \leqslant \alpha < \beta \leqslant N} \cos(\lambda_\alpha - \lambda_\beta) \prod_{1 \leqslant k < j \leqslant N} \cos(\nu_k - \nu_j). \tag{58}$$

For the correlation functions $H_N^{(M)}$ and $G_N^{(M)}$ of the model in the free fermion case it is worth using directly representations (34) and (38). Substituting formula (58) in representation (34), one gets for the correlation function $H_N^{(M)}$,

$$\begin{aligned}
H_N^{(M)} = {} & \frac{\prod_{\alpha=1}^{M-1} \sin\left(\lambda_\alpha - \nu_1 - \tfrac{\pi}{4}\right) \prod_{\alpha=M+1}^N \sin\left(\lambda_\alpha - \nu_1 + \tfrac{\pi}{4}\right)}{\prod_{j=1}^N \cos(\nu_j - \nu_1)} \\
& \times \sum_{\beta=1}^M \frac{\prod_{j=2}^N \sin\left(\lambda_\beta - \nu_j + \tfrac{\pi}{4}\right)}{\prod_{\alpha=M}^N \cos(\lambda_\alpha - \lambda_\beta)} \prod_{\substack{\alpha=1 \\ \alpha \neq \beta}}^M \frac{1}{\sin(\lambda_\alpha - \lambda_\beta)}.
\end{aligned} \tag{59}$$

A similar formula can easily be written for the correlation function $G_N^{(M)}$.

Consider the homogeneous limit of the correlation functions in the free fermion case. The homogeneous model is obtained by putting the variables in each set $\{\lambda_\alpha\}_{\alpha=1}^N$ and $\{\nu_k\}_{k=1}^N$ to be equal:

$$\lambda_1 = \cdots = \lambda_N \equiv \lambda \qquad \nu_1 = \cdots = \nu_N \equiv \nu. \tag{60}$$

Since now all the vertex weights depend only on the difference $\lambda - \nu$, without loss of generality one can take $\nu = -\frac{\pi}{4}$. Thus, we have the homogeneous model with the following parametrization of the vertex weights:

$$\mathsf{a}(\lambda) = \cos\lambda \qquad \mathsf{b}(\lambda) = \sin\lambda \qquad \mathsf{c}(\lambda) = 1. \tag{61}$$

Note that the partition function Z_N of the homogeneous model is independent of λ and is equal to 1, $Z_N = 1$, see (58).

The substitution $\nu_k = \nu = -\frac{\pi}{4}$ in expression (59) leads us to the following intermediate expression for $H_N^{(M)}$, when all ν_k are equal while all λ_α are still different:

$$H_N^{(M)} = \prod_{\alpha=1}^{M-1} \sin\lambda_\alpha \prod_{\alpha=M+1}^N \cos\lambda_\alpha \sum_{\beta=1}^M \frac{\cos^{N-1}\lambda_\beta}{\prod_{\alpha=M}^N \cos(\lambda_\alpha - \lambda_\beta)} \prod_{\substack{\alpha=1 \\ \alpha \neq \beta}}^M \frac{1}{\sin(\lambda_\alpha - \lambda_\beta)}. \tag{62}$$

The problem now is to obtain the limit of expression (62) when all λ_α tend to the same value. For this purpose it is convenient to rewrite (62) in terms of rational functions instead of trigonometric ones:

$$H_N^{(M)} = \left(1 + u_M^2\right) \prod_{\alpha=1}^{M-1} u_\alpha \sum_{\beta=1}^{M} \prod_{\alpha=M}^{N} \frac{1}{1 + u_\alpha u_\beta} \prod_{\substack{\alpha=1 \\ \alpha \neq \beta}}^{M} \frac{1}{u_\alpha - u_\beta} \tag{63}$$

where

$$u_\alpha = \tan \lambda_\alpha \qquad \alpha = 1, \ldots, N. \tag{64}$$

Therefore, one should take the homogeneous limit in the set $\{u_\alpha\}_{\alpha=1}^{N}$:

$$u_\alpha \to u \qquad \alpha = 1, \ldots, N. \tag{65}$$

For the subset $\{u_\alpha\}_{\alpha=M}^{N}$ we may simply put $u_M = u_{M+1} = \cdots = u_N = u$, while for the subset $\{u_\alpha\}_{\alpha=1}^{M-1}$ we parametrize u_α as

$$u_\alpha = u - (M - \alpha)\varepsilon \qquad \alpha = 1, \ldots, M - 1 \tag{66}$$

and the homogeneous limit then corresponds to the case of vanishing ε. Since the prefactor of the sum over β in (63) is regular as $\varepsilon \to 0$, one may put $\varepsilon = 0$ in this prefactor. Thus, for $H_N^{(M)}$ one gets

$$H_N^{(M)} = (1 + u^2) u^{M-1} \sum_{\beta=1}^{M} \frac{1}{[1 + u^2 - u(M-\beta)\varepsilon]^{N-M+1} \varepsilon^{M-1}} \prod_{\substack{\alpha=1 \\ \alpha \neq \beta}}^{M} \frac{1}{\alpha - \beta}$$

$$= (1 + u^2) u^{M-1} \sum_{\beta=0}^{M-1} \frac{(-1)^{M-1}}{(1 + u^2 - u\beta\varepsilon)^{N-M+1} \varepsilon^{M-1}} \prod_{\substack{\alpha=1 \\ \alpha \neq \beta}}^{M-1} \frac{1}{\alpha - \beta}. \tag{67}$$

Taking into account that

$$\prod_{\substack{\alpha=0 \\ \alpha \neq \beta}}^{M-1} \frac{1}{\alpha - \beta} = \frac{(-1)^\beta}{\beta!(M - \beta - 1)!} = \frac{(-1)^\beta}{(M-1)!} \binom{M-1}{\beta} \tag{68}$$

one obtains

$$H_N^{(M)} = \frac{(-1)^{M-1}(1 + u^2)}{(M-1)! u^{N-2M+2}} \sum_{\beta=0}^{M-1} \frac{(-1)^\beta}{[(1+u^2)u^{-1} - \varepsilon\beta]^{N-M+1} \varepsilon^{M-1}} \binom{M-1}{\beta}. \tag{69}$$

The sum over β in (69) in the limit $\varepsilon \to 0$ becomes exactly the $(M-1)$th derivative of a function $f(z)$ with respect to a variable z:

$$\lim_{\varepsilon \to 0} \sum_{\beta=0}^{M-1} \frac{(-1)^\beta f(z - \beta\varepsilon)}{\varepsilon^{M-1}} \binom{M-1}{\beta} = \frac{\mathrm{d}^{M-1}}{\mathrm{d}z^{M-1}} f(z) \tag{70}$$

where

$$f(z) = \frac{1}{z^{N-M+1}} \qquad z = \frac{1 + u^2}{u}. \tag{71}$$

Hence, for the sum over β in (69) in the limit $\varepsilon \to 0$ one gets

$$\lim_{\varepsilon \to 0} \sum_{\beta=0}^{M-1} \frac{(-1)^\beta}{[(1+u^2)u^{-1} - \varepsilon\beta]^{N-M+1} \varepsilon^{M-1}} \binom{M-1}{\beta} = \frac{(-1)^{M-1}(N-1)!}{(N-M)!} \left(\frac{u}{1+u^2}\right)^N. \tag{72}$$

Finally, substituting expression (72) in (69) and taking into account that $u = \tan\lambda$, we obtain the following expression for the correlation function $H_N^{(M)}$ of the homogeneous six-vertex model in the free fermion case (61):

$$H_N^{(M)} = \binom{N-1}{M-1}(\cos^2\lambda)^{N-M}(\sin^2\lambda)^{M-1}. \tag{73}$$

Similarly, for the correlation function $G_N^{(M)}$ one gets

$$G_N^{(M)} = \sum_{K=1}^{M}\binom{N-1}{K-1}(\cos^2\lambda)^{N-K}(\sin^2\lambda)^{K-1}. \tag{74}$$

Obviously, connection formulae (26) and (27) are fulfilled by (73) and (74).

It is clear from the results obtained that the boundary correlation functions significantly depend on both λ and M even in the simplest free fermion case. This dependence of the function $G_N^{(M)}$ is of special interest in the thermodynamic limit when both N and M go to infinity so that the variable $x = M/N$ runs through the interval $0 < x < 1$. Let us denote the function $G_N^{(M)}$ in this limit as $G(x)$. Representing (74) in the form

$$G_N^{(M)} = \binom{N-1}{M-1}(\sin^2\lambda)^{M-1}(\cos^2\lambda)^{N-M}\,_2F_1(1, -M+1; N-M+1; -\cot^2\lambda) \tag{75}$$

and using the integral representation for the hypergeometric function $_2F_1$, one gets

$$G_N^{(M)} = \frac{(N-1)!}{(N-M-1)!(M-1)!}(\sin^2\lambda)^{M-1}(\cos^2\lambda)^{N-M}$$
$$\times \int_0^1 \mathrm{d}t\,(1-t)^{N-M-1}(1+t\cot^2\lambda)^{M-1} \qquad M = 1,\ldots,N-1. \tag{76}$$

Applying the steepest descent method to representation (76) we obtain that $G(x)$ is the Heaviside step function:

$$G(x) = \theta(x - \sin^2\lambda) \qquad \theta(\xi) = \begin{cases} 1 & \text{if } \xi > 0 \\ \frac{1}{2} & \text{if } \xi = 0 \\ 0 & \text{if } \xi < 0. \end{cases} \tag{77}$$

This result means that the arrows at the boundary column are ordered (frozen) in the thermodynamic limit: all arrows are pointing down above the point $x = \sin^2\lambda$, while below all of them are pointing up.

6. Determinant representations in the homogeneous limit

In this section the homogeneous limit of determinant representations (50) and (53) for the boundary correlation functions is considered in the general case. To obtain the homogeneous model one should put all λ_α equal to λ and all ν_k equal to ν in the inhomogeneous model (1). Without loss of generality one may also put $\nu = 0$. Hence, the vertex weights of the homogeneous six-vertex model are given by

$$\mathsf{a}(\lambda) = \sinh(\lambda + \eta) \qquad \mathsf{b}(\lambda) = \sinh(\lambda - \eta) \qquad \mathsf{c}(\lambda) = \sinh 2\eta. \tag{78}$$

The procedure of taking the homogeneous limit for the partition function Z_N has been elaborated in the papers [17, 18]. In this limit, the singularities coming from the denominator of expression (21) are cancelled by the zeros coming from the determinant since then all rows and columns of the matrix $\mathcal{Z}$ tend to each other.

It can be shown by Taylor expansion of the entries of the matrix $\mathcal{Z}$ that the partition function of the homogeneous model is expressed through the double Wronskian:

$$Z_N = \frac{\det_N \mathcal{Z}_{\mathrm{hom}}}{[\phi(\lambda)]^{N^2} \prod_{n=1}^{N-1} (n!)^2} \qquad (\mathcal{Z}_{\mathrm{hom}})_{\alpha k} = \frac{\mathrm{d}^{\alpha+k-2}}{\mathrm{d}\lambda^{\alpha+k-2}} \phi(\lambda) \tag{79}$$

where $\phi(\lambda) \equiv \phi(\lambda, 0)$, namely,

$$\phi(\lambda) = \frac{\sinh 2\eta}{\sinh(\lambda + \eta) \sinh(\lambda - \eta)}. \tag{80}$$

Since the correlation functions have been expressed in equations (50) and (53) through determinants, one may apply the approach described in detail in [18] to obtain the correlation functions of the homogeneous model.

To apply the procedure given in [18] with minimal modifications, it is convenient to consider the function $\tilde{G}_N^{(M)}$ instead of $G_N^{(M)}$, where

$$\tilde{G}_N^{(M)}(\lambda_1, \ldots, \lambda_N; \nu_1, \ldots, \nu_N) = G_N^{(M)}(\lambda_N, \ldots, \lambda_1; \nu_N, \ldots, \nu_1). \tag{81}$$

Clearly, these functions are equal in the homogeneous limit. Comparing equations (53) and (81) one can see that the function $\tilde{G}_N^{(M)}$ may be written as

$$\tilde{G}_N^{(M)} = \frac{\prod_{k=1}^{N-1} \sinh(\nu_N - \nu_k)}{\prod_{\alpha=N-M+1}^{N} \sinh(\lambda_\alpha - \nu_N + \eta) \prod_{\alpha=1}^{N-M} \sinh(\lambda_\alpha - \nu_N - \eta)} \frac{\det_N \tilde{\mathcal{G}}}{\det_N \mathcal{Z}} \tag{82}$$

where

$$\tilde{\mathcal{G}}_{\alpha k} = \phi(\lambda_\alpha, \nu_k) \qquad \tilde{\mathcal{G}}_{\alpha N} = \tilde{g}_M(\lambda_\alpha) \qquad k = 1, \ldots, N-1 \tag{83}$$

and the function $\tilde{g}_M(\lambda)$ is defined as

$$\tilde{g}_M(\lambda) = \frac{\prod_{\gamma=N-M+1}^{N} \sinh(\lambda_\gamma - \lambda + 2\eta) \prod_{\gamma=1}^{N-M} \sinh(\lambda_\gamma - \lambda)}{\prod_{k=1}^{N} \sinh(\lambda - \nu_k - \eta)}. \tag{84}$$

The matrix $\tilde{\mathcal{G}}$ differs from the matrix $\mathcal{Z}$ by the last column, and the first $N - M$ entries in the last column of the matrix $\tilde{\mathcal{G}}$ are equal to zero, $\tilde{\mathcal{G}}_{N1} = \cdots = \tilde{\mathcal{G}}_{N N-M} = 0$.

Now, the homogeneous limit in the set $\{\lambda_\alpha\}_{\alpha=1}^N$ can be easily found following [18]. Representing the differences $\lambda_\gamma - \lambda_\alpha$ in the expression for the quantity $\tilde{g}_M(\lambda_\alpha)$ as $(\lambda_\gamma - \lambda) - (\lambda_\alpha - \lambda)$, one may consider the differences $\lambda_\alpha - \lambda$ as independent variables. In the limit $\lambda_\alpha \to \lambda$, $\alpha = 1, \ldots, N$, these variables tend to zero and it can be proved that the entries in the last column of the matrix $\mathcal{Z}$, after successive subtractions of the rows, become the coefficients in the Taylor expansion of the function

$$\psi(\varepsilon) = (-1)^N \frac{(\sinh \varepsilon)^{N-M} \sinh^M(\varepsilon - 2\eta)}{\prod_{j=1}^{N} \sinh(\varepsilon + \lambda - \nu_j - \eta)} \tag{85}$$

at the point $\varepsilon = 0$. This solves the problem of taking the homogeneous limit in the set $\{\lambda_\alpha\}_{\alpha=1}^N$. The homogeneous limit in the set $\{\nu_k\}_{k=1}^N$ can be taken in the same manner as for the partition function, with the only difference that there are no subtractions from the last column; one should simply put all ν_k equal to $\nu = 0$ in the entries of the last column.

As a result, one obtains the following determinant representation for the correlation function $G_N^{(M)}$ in the homogeneous limit:

$$G_N^{(M)} = \frac{(N-1)!}{[\sinh(\lambda + \eta)]^M [\sinh(\lambda - \eta)]^{N-M}} \frac{\det_N \mathcal{G}_{\mathrm{hom}}}{\det_N \mathcal{Z}_{\mathrm{hom}}} \tag{86}$$

where

$$(\mathcal{G}_{\text{hom}})_{\alpha k} = (\mathcal{Z}_{\text{hom}})_{\alpha k} \qquad k = 1, \ldots, N-1$$

$$(\mathcal{G}_{\text{hom}})_{\alpha N} = -\frac{\mathrm{d}^{\alpha-1}}{\mathrm{d}\varepsilon^{\alpha-1}} \left\{ \frac{(\sinh \varepsilon)^{N-M}[\sinh(\varepsilon - 2\eta)]^M}{[\sinh(\varepsilon + \lambda - \eta)]^N} \right\} \Bigg|_{\varepsilon=0}. \qquad (87)$$

For the correlation function $H_N^{(M)}$ in the homogeneous limit the following expression is valid:

$$H_N^{(M)} = \frac{(N-1)! \sinh 2\eta}{[\sinh(\lambda + \eta)]^M [\sinh(\lambda - \eta)]^{N-M+1}} \frac{\det_N \mathcal{H}_{\text{hom}}}{\det_N \mathcal{Z}_{\text{hom}}} \qquad (88)$$

where

$$(\mathcal{H}_{\text{hom}})_{\alpha k} = (\mathcal{Z}_{\text{hom}})_{\alpha k} \qquad k = 1, \ldots, N-1$$

$$(\mathcal{H}_{\text{hom}})_{\alpha N} = \frac{\mathrm{d}^{\alpha-1}}{\mathrm{d}\varepsilon^{\alpha-1}} \left\{ \frac{(\sinh \varepsilon)^{N-M}[\sinh(\varepsilon - 2\eta)]^{M-1}}{[\sinh(\varepsilon + \lambda - \eta)]^{N-1}} \right\} \Bigg|_{\varepsilon=0}. \qquad (89)$$

Formulae (86) and (88) generalize representation (79) for the partition function of the homogeneous model, and they may be used for the investigation of the correlation functions in the thermodynamic limit.

7. Conclusion

In the present paper the determinant representation for the one-point boundary correlation functions of the six-vertex model with the domain wall boundary conditions is derived. For the correlation functions H_N^M (25) and $G_N^{(M)}$ (24) we have obtained representations (50) and (53), which generalize the determinant representation for the partition function Z_N. In these formulae the correlation functions are expressed through the determinants of the $N \times N$ matrices $\mathcal{H}$ and $\mathcal{G}$, respectively. The matrices $\mathcal{H}$ and $\mathcal{G}$ differ from the matrix $\mathcal{Z}$ appearing in representation (21) for the partition function by the first column only. The derivation of representations (50) and (53) is based on reduction formulae (34) and (38) which have been obtained in section 3 exclusively by means of the algebra of operators entering the monodromy matrix.

In the free fermion case studied in section 5 we have obtained explicit (determinant-free) formulae (73) and (74) for the boundary correlation functions H_N^M and $G_N^{(M)}$ of the homogeneous model. In the thermodynamic limit $N, M \to \infty$ the function $G_N^{(M)}$ describing the spontaneous polarization at the boundary turns into the function $G(x)$ with $x = M/N$, $0 < x < 1$. We have found that $G(x)$ is just the Heaviside step function (77). The emergence of the step function for the spontaneous polarization indicates the 'freezing' of the arrows at the boundary in the thermodynamic limit. On the other hand, from the results of the paper [20] we have found numerically that at the ice point, $\mathsf{a} = \mathsf{b} = \mathsf{c} = 1$, the function $G(x)$ exhibits a similar behaviour, $G(x) = \theta(x - 1/2)$. The obtained determinant formulae (86) and (88) may be used for the investigation of the boundary correlations in the thermodynamic limit in the homogeneous model with arbitrary values of the vertex weights.

The step function behaviour of the boundary spontaneous polarization indicates the existence of the analogue of the arctic circle theorem [21, 22]. To be more precise, we expect that for the values $-1 < \cosh 2\eta < 1$ the arrows are 'frozen' at the corners of the grid, while inside the grid there is a region of 'disorder'. To obtain the shape of the 'disordered' region of the grid it is necessary to obtain an appropriate expression for the spontaneous polarization not only at the boundary but also at an arbitrary point of the lattice. Hence, the problem of the calculation of the correlation functions of the model deserves further investigation. We hope that the approach described in sections 3 and 4 may appear to be productive in the derivation of the correlation functions outside the boundary.

Acknowledgments

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